4/9/25 1hr

18:20 - 19:20pm

$\frac{4}{5} = 80\%$

1. Compute the following integral:

$$\int_1^4 \sqrt{t}\,\ln t\,dt$$

$$\int_1^4 \sqrt{t}\ln t\,dt$$

$$= \ln t \cdot \frac{2}{3}t^{3/2}\Big|_1^4 - \int_1^4 \frac{1}{t}\cdot\frac{2}{3}t^{3/2}\,dt$$

$$= \frac{2t^{3/2}\ln t}{3} - \frac{2}{3}\left(\frac{2}{3}t^{3/2}\right)\Big|_1^4$$

$$= \frac{2}{3}\left(t^{3/2}\left(\ln t - \frac{2}{3}\right)\right)\Big|_1^4$$

$$= \frac{2}{3}\left(8\left(\ln 4 - \frac{2}{3}\right) - \left(-\frac{2}{3}\right)\right)$$

$$= \frac{2}{3}\left(8\ln 4 - \frac{14}{3}\right)$$

$u = \ln t$

$v' = \sqrt{t}$

$\Rightarrow u' = \frac{1}{t}$

$v = \frac{2}{3}t^{3/2} + C$

2. Compute the following integral:

$$\int_0^{\pi/4} \tan^4\theta \sec^6\theta \, d\theta$$

$u = \tan\theta$

$du = \sec^2\theta \, d\theta$

$u = \tan 0 = 0$

$u = \tan\frac{\pi}{4} = 1$

$$\int \tan^4\theta \sec^6\theta \, d\theta$$

$$= \int \tan^4\theta \sec^4\theta \sec^2\theta \, d\theta$$

$$= \int \tan^4\theta \sec^2\theta \left(1+\tan^2\theta\right)^2 d\theta$$

$$= \int \tan^4\theta \sec^2\theta + 2\tan^6\theta \sec^2\theta + \tan^8\theta \sec^2\theta \, d\theta$$

$$= \int u^4 + 2u^6 + u^8 \, du$$

$$= \frac{u^5}{5} + \frac{2u^7}{7} + \frac{u^9}{9} + C$$

$$\int_0^{\pi/4} \tan^4\theta \sec^6\theta \, d\theta$$

$$= \left.\frac{u^5}{5} + \frac{2u^7}{7} + \frac{u^9}{9}\right|_0^1$$

$$= \frac{1}{5} + \frac{2}{7} + \frac{1}{9}$$

3. Compute the following integral:

$$\int \frac{10}{(x-1)(x^2+9)}\,dx$$

$$\int \frac{10}{(x-1)(x^2+9)}\,dx$$

$$= \int \frac{A}{x-1} + \frac{Bx+C}{x^2+9}\,dx$$

$$= \int \frac{1}{x-1} + \frac{x-1}{x^2+9}\,dx$$

$$= \ln|x-1| + \frac{1}{2}\ln|x^2+9| - \frac{\arctan\frac{x}{3}}{3}$$

$$+C$$

$$x=1$$

$$\Rightarrow \frac{10}{1^2+9} = A$$

$$A = 1$$

$$10 = A(x^2+9) + (Bx+C)(x-1)$$

$$x=0 \Rightarrow 10 = 9A + C(-1)$$

$$C = 9-10$$

$$= -1$$

$$Bx + Cx = 0$$

$$\Rightarrow B + C = 0$$

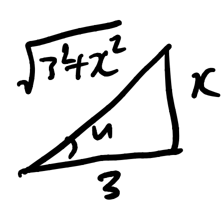
$$B = 1$$

$$3\tan u = x$$

$$dx = 3\sec^2 u\,du$$

$$\int \frac{x-1}{x^2+9}\,dx$$

$$= \int \frac{(3\tan u - 1)\,\cancel{3\sec^2 u}}{9\,\cancel{(\tan^2 u + 1)}}\,du$$



$$= \frac{1}{3}\int 3\tan u - 1\,du$$

$$= \frac{1}{3}\left(-3\ln|\cos u| - u\right) + C$$

$$= -\ln 3 + \frac{\ln(x^2+9)}{2} - \frac{\arctan\frac{x}{3}}{3} + C$$

4. Compute the following integral:

$$\int \frac{1}{(5-4x-x^2)^{5/2}}\,dx$$

4. $\displaystyle\int \frac{1}{(5-4x-x^2)^{5/2}}\,dx$

$\displaystyle= \int \frac{1}{(-(x+2)^2+9)^{5/2}}\,dx$

$\displaystyle= \int \frac{1}{(-u^2+9)^{5/2}}\,du$

$\displaystyle= \int \frac{3\cos\theta\, d\theta}{(-9\sin^2\theta+9)^{5/2}}$

$\displaystyle= \frac{1}{3^4}\int \frac{\cos\theta\, d\theta}{(1-\sin^2\theta)^{5/2}}$

$\displaystyle= \frac{1}{81}\int \frac{\cos\theta\, d\theta}{\cos^5\theta}$

$\displaystyle= \frac{1}{81}\int \frac{1}{\cos^4\theta}\,d\theta$ $\quad \frac{1}{2}$

$\displaystyle= \frac{1}{81}\int \frac{8}{1+\cos 4\theta}\,d\theta$ $\quad \times$

$5-4x-x^2$

$= -(x^2+4x-5)$

$= -(x^2+4x+2^2-2^2-5)$

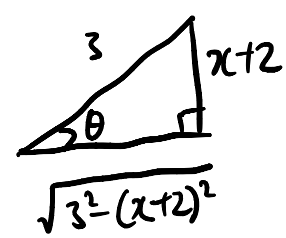
$= -(x+2)^2+9$

Let $u=x+2$. $du=dx$

$u=3\sin\theta$, $du=3\cos\theta\, d\theta$

$\sin\theta = \dfrac{x+2}{3}$

$\theta = \sin^{-1}\left(\dfrac{x+2}{3}\right)$



$\cos^4\theta = (1-\sin^2\theta)\cos^2\theta$

$\displaystyle= \left(1-\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)$

$\displaystyle= \frac{1-\cos^2 2\theta}{4}$

$\displaystyle= \frac{1}{4} - \frac{1+\cos 4\theta}{8}$

$\displaystyle= \frac{1}{4} - \frac{1}{8} + \frac{\cos 4\theta}{8}$

$\displaystyle= \frac{1+\cos 4\theta}{8}$

5. (a) Set up (but do not solve) the integral for the arc length along the curve $x = y + y^3$ from $y = 1$ to $y = 4$.

$$ds^2 = dx^2 + dy^2$$

$$\frac{dx}{dy} = 1 + 3y^2$$

$$\Rightarrow \frac{ds}{dy} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}$$

$$= \sqrt{1 + 6y^2 + 9y^4 + 1}$$

$$= \sqrt{2 + 6y^2 + 9y^4}$$

$$\Rightarrow s = \int_1^4 \sqrt{2 + 6y^2 + 9y^4}\, dy$$

Solution answer:

$$\int ds = \int_1^4 \sqrt{(1+3t)^2 + 1}\, dt$$

(b) Set up (but do not solve) the integral for the surface area of the surface obtained by rotating the curve given by

$$x = a\cos^3 t, \quad y = a\sin^3 t, \quad 0 \leqslant t \leqslant \pi/2$$

about the x-axis. Here a is an arbitrary constant.

$$x^{2/3} + y^{2/3} = a^{2/3}(\cos^2 t + \sin^2 t)$$

$$x^{2/3} + y^{2/3} = a^{2/3}$$

$$\frac{dx}{dt} = 3a\cos^2 t(-\sin t)$$

$$\frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= 3a\sqrt{\cos^4 t\sin^2 t + \sin^4 t\cos^2 t}$$

$$= 3a\cos t\sin t$$

0.5

$2\pi y$

$$dS = 2\pi x\, ds$$

$$= 2\pi(a\cos^3 t)\, 3a\cos t\sin t\, dt$$

$$S = 6a^2\pi \int_0^{\frac{\pi}{2}} \cos^4 t\sin t\, dt$$